Providing Opportunities to Develop Prospective Teachers' Pedagogical Content Knowledge

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Abstract: *One responsibility of mathematics teacher educators is to provide opportunities for prospective teachers to develop the necessary knowledge to be highly qualified professionals and effective mathematics teachers. Hill, Ball, and Schilling (2008) delineated two major knowledge domains required for mathematics teachers to be effective: subject matter knowledge and pedagogical content knowledge. In this article, based on literature review, we explore specific recommendations for mathematics teacher educators, including mathematicians and novice mathematics teacher educators, on different ways to incorporate lesson ideas into mathematics content courses to help provide opportunities for grades K-8 (age 5-14) prospective teachers' development of pedagogical content knowledge. The work here provides a foundation for mathematics teacher educators' practice, encompassing classroom-based examples and strategies, and offers potential avenues for mathematics teacher educators to conduct their own research on the nature and effectiveness of their course activities that help to promote and develop prospective teachers' pedagogical content knowledge.*

Key words: pedagogical content knowledge, mathematics teacher education, prospective teachers

Introduction

Recent research and political initiatives have suggested that while in teacher education programs, many prospective teachers (PTs) receive inadequate preparation to become effective teachers of mathematics, particularly related to the development of deep and profound knowledge of *mathematics* and *pedagogy* required for teaching children mathematics (e.g., Conference Board of the Mathematical Sciences (CBMS), 2012; Greenberg & Walsh, 2008). These issues stem from, among other things, a lack of clarity and consensus within the field

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about what PTs should learn during their university mathematics education experiences, particularly in mathematics content courses (e.g., Ball, Sleep, Boerst, & Bass, 2009; Zaslavsky, 2007), and a lack of preparation and support for the mathematics teacher educators (MTEs) who teach content courses for PTs (e.g., Bergsten & Grevholm, 2008).

Studies show that developing *pedagogical content knowledge* and *subject matter knowledge* is critical for PTs' development (An, Kulm, & Wu, 2004; Blömeke, Buchholtz, Suhl, & Kaiser, 2014). Pedagogical content knowledge (PCK) is broadly defined in the literature as the knowledge of effective teaching which includes, but is not limited to, teachers' deep and conceptual knowledge of broad and specific mathematical topics for the grade level taught, knowledge of curriculum that enables teachers to effectively use (and select) curriculum materials, knowledge of teaching and instructional strategies as well as the knowledge of students' learning to be able to effectively plan lessons and prepare instruction to target students' conceptions and misconceptions and address their learning needs (e.g., An et al., 2004; Blömeke, et al., 2014). In contrast, subject matter knowledge also reflects teachers' knowledge needed for teaching mathematics, however, it includes elements that do not depend on knowledge of students or teaching. Subject matter knowledge is comprised of three domains: 1) common content knowledge of mathematics used [commonly] in the teaching profession and other occupations, 2) specialized content knowledge that allows teachers to "engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill, Ball, & Schilling, 2008, p. 377-378), and 3) mathematical knowledge at the horizon which reflects the developments within the field. Hill et al. (2008) argue that having knowledge of these two specific domains is critical for mathematics teachers to be effective. Moreover, An et al. (2004) argue PCK "has a deep impact on teaching practice" (p. 169).

There has been an increase in the efforts of MTEs to study and share the work they are doing with PTs regarding both the content (e.g., Ball, et al., 2009) and pedagogy (e.g., Lampert et al., 2013; Ghousseini & Herbst, 2016). However, much of this work is documented in the mathematics methods courses, and very little is known about MTEs' work in the content courses, particularly the development of PTs' PCK (e.g., An, Kulm, & Wu, 2004; Bergsten & Grevholm, 2008; Blömeke, et al., 2014; Chick & Beswick, 2017). Research has indicated that mathematics content courses are often taught by MTEs in mathematics departments who have little to no experience working with elementary/middle level content or K-8 students (ages 6-14) and do not receive the training or support necessary to effectively address the needs of the PTs with whom they work (Masingila, Olanoff, & Kwaka, 2012). Similarly, the authors of *The Mathematical Education of Teachers* (MET) *II* report concluded that future K-8 teachers, like many undergraduates, enter college with a "superficial knowledge of K-12 mathematics" and very little understanding of the K-8 mathematics they will teach (CBMS, 2012, p. 12). Furthermore, the authors of the MET II report state,

> much that is useful to (K-8) teachers is known about teaching and learning paths for early mathematics, but, too often, mathematicians who are new to this area lack the knowledge or resources to help future teachers develop an understanding of these paths … (p. 4)

In other words, MTEs who teach mathematics content courses are involved in providing learning opportunities that directly shape and influence PTs' knowledge and readiness to teach, including PCK (Hallett, Nunes, & Bryant, 2010; Hiebert & Lefevre, 2013; Thanheiser et al., 2014).

However, evidence suggests that when teaching mathematics content courses, MTEs predominantly focus on subject matter knowledge, largely circumventing PTs' development of PCK in the content courses (Ball, 1988; Schmidt, Burroughs, Cogan, & Houang, 2017; Laursen, Hassi, & Hough, 2015). This is especially problematic because the development of PCK is passed up as a learning opportunity during the content courses and is kept untapped and static until the mathematics methods course, which PTs typically complete the semester prior to student teaching. Researchers argue that one methods course is not enough for PTs to develop the necessary PCK, including pedagogical expertise and knowledge about instructional strategies for teaching mathematics and skills necessary to be able to anticipate and address children's mathematical conceptions and misconceptions (Greenberg & Walsh, 2008; Lutzer, Rodi, Kirkman, & Maxwell, 2007).

Additionally, since most teacher education programs require PTs to complete mathematics content courses (CBMS, 2001; Greenberg & Walsh, 2008; Masingila et al., 2012), specifically designed to enhance PTs' knowledge by having them make sense of concepts and principles that underlie the mathematics they learned as children (and will teach to children), mathematics content courses may offer an ideal platform for initiating PTs' learning experiences with PCK (Ambrose, 2004; Bass, 2005; Wideen, Mayer-Smith, & Moon, 1998). Thus, in this manuscript we explore specific recommendations from literature that can help MTEs, including mathematicians and novice MTEs, incorporate lesson ideas into their K-8 content courses in order to provide them with opportunities to develop K-8 PTs' PCK. We define "content courses" as courses specifically designed for K-8 PTs focused around learning the mathematics content they will teach. Studies report that these courses are typically taught in the mathematics department, completed by PTs in their first couple of years of their undergraduate program, and mainly focus on strengthening PTs' knowledge of mathematical topics they will teach. These courses mainly comprise topics in number and operations, geometry and measurement, and number theory (specifically, number and operations [95%], geometry [91%], measurement

[88%], number theory [87%], probability [82%], algebra [80%], and statistics [77%]), however they largely focus on "content" and not "pedagogy" related to teaching mathematics to K-8 children (Greenberg & Walsh, 2008; Lutzer et al., 2007; Masingila et al., 2012).

Theoretical Framework

Pedagogical content knowledge (PCK) has been identified as a special type of teachers' knowledge that "represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interest and abilities of learners, and presented for instruction" (Shulman, 1987, p. 8). Hill et al. (2008) built on Shulman's work and identified three central domains of PCK: knowledge of content and teaching, knowledge of curriculum, and knowledge of content and students. For the purpose of this paper, we define *knowledge of content and teaching* as teachers' knowledge about specific instructional methods, activities, manipulatives, and different grade-level strategies to teaching specific math concepts (Ball, Thames, & Phelps, 2008). *Knowledge of curriculum* includes overall goals, objectives, and programs as well as specific curricular materials, standards, resources relevant (and effective) to teaching mathematics content, and scope and sequence of mathematical topics across grade levels and interdisciplinary connections of mathematics to other subjects (Shulman, 1987). *Knowledge of content and students* involves the teachers' knowledge about specific approaches and strategies that help to responsibly and responsively address students' specific mathematical conceptions and misconceptions (Hill et al., 2008).

A substantial body of research suggests that learning opportunities specifically designed around PCK development help strengthen PTs' knowledge about mathematics they will teach, as well as their ability to critically reflect on their classroom instruction and improve student learning (e.g., Baumert et al., 2010; Burton et al., 2008; Capraro et al., 2005; Carpenter et al., 1988; Tirosh et al., 2011; Vale, 2010). Additionally, increasing and deepening PCK is critical for the development of PTs' professional identity and readiness to teach (An et al., 2004; Blömeke et al., 2014). Therefore, in this manuscript, we address various ways/practitioner approaches that MTEs can incorporate in their content courses in order to provide PTs with opportunities to develop PCK.

About This Study

We conducted a basic "classification systems" literature review (Webster & Watson, 2002), in which we used categories to review, classify, and summarize observations from documented empirical evidence related to our main research question: *What lesson ideas and practitioner recommendations are documented in the literature on MTEs' practices for providing opportunities for K-8 PTs to develop PCK in mathematics content courses*? We searched the mathematics education literature, published in the last two decades (1998-2018), to answer the research question. Below, we describe our procedures. We would like to emphasize (and remind the reader) that our literature review focused specifically on the reported work of MTEs in K-8 mathematics content courses.

We conducted a Boolean literature search, within multiple databases and journals, combing through the results and manually identifying relevant literature, including literature searches for each PCK domain separately (i.e., knowledge of instructional strategies, knowledge of curriculum, and knowledge of student understanding). We used "and/or" combinations of different keywords (e.g., content, content courses, mathematics teacher educators, pedagogical content knowledge, preservice teachers, prospective teachers, curriculum knowledge, instructional strategies, instructional practices, student learning, children learning). From these searches, we selected the scholarly products that were of explicit practitioner nature, referencing teaching mathematical content to PTs and including examples, implications, and evidence of specific learning opportunities designed to provide opportunities to develop PTs' PCK. We read the abstracts of these articles to identify specific ones for full review.

The final number of the scholarly products reviewed for this study was 71. Most of them included practitioner articles published by the National Council of Teachers of Mathematics (NCTM) journals (e.g., *Mathematics Teacher Educator, Mathematics Teaching in the Middle School, Teaching Children Mathematics*), as well as books and book chapters, monographs, and other mathematics education articles published in journals that often included practice-based recommendations, such as (but not limited to): (a) *Journal of Mathematics Teacher Education*, (b) *School Science and Mathematics*, and (c) *International Journal of Educational Research*.

Due to a smaller sample size of the articles, we conducted a qualitative *thematic* literature review of these selected scholarly products, identifying common themes/trends in the literature through open coding (Corbin & Strauss, 2008), to provide a comprehensive summary of specific (documented) classroom-based examples of PCK learning opportunities afforded by MTEs when teaching content to PTs. For example, when reviewing literature related to the opportunities afforded by MTEs to PTs related to the *knowledge of students*, we identified two major themes: developing knowledge of students via PTs' *direct* and *indirect* interactions with them. Furthermore, within the "direct interactions" literature, we also identified several clusters of articles situated around a specific focus: interviews, quick written assessments, family nights and afterschool projects, and math pen-pal activities. Similarly, within the "indirect interactions" literature, we identified several clusters of articles addressing one of the following foci: transcripts and videos, correct and incorrect written students' work, and hypothetical situations that required PTs' responses to children's mathematics. Below, we describe specific examples from the literature and provide quotes and descriptions directly from the articles as helpful (representative) and practical exemplars. We do not claim that this review of the literature is a "know how" that provides MTEs with specific ways on how to implement these examples in their classrooms. In fact, we suggest that readers who come across a specific example they would like to facilitate in their classroom delve deeper into the article(s) cited under that example to learn more about the implementation for their classroom. We acknowledge this review may not be exhaustive, but it is a start to synthesizing the literature related to MTEs providing opportunities to develop PTs' PCK in K-8 mathematics content courses.

Opportunities to Develop Pedagogical Content Knowledge

Opportunities to Develop Prospective Teachers' Knowledge of Content and Teaching

For more than a decade, researchers have reported results from studies that have identified various teaching strategies* that teachers employ in K-8 classrooms. Some of the identified strategies are specific to a certain context (e.g., different curricula, collaborative learning environments) while others are more general and geared towards helping teachers enhance and improve their overall teaching practice (e.g., classroom management, creating collaborative and safe learning environments). Below, we describe the most prevalent images of four themes found in the literature that MTEs employ in mathematics content courses to promote PTs' knowledge of content and teaching mathematics to K-8 students. The four themes are: 1)

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*Some researchers use the term "instructional strategies" (House & Telese, 2008; Rieg & Wilson, 2009; Staples, 2007; Weiss, Pasley, Smith, Banilower, & Heck, 2003) while others use the term "instructional practices" (Drummond, 2002; Herbel-Eisenmann, Lubienski, & Id-Deen, 2006; Newton, 2009).

implementing worthwhile tasks; 2) using physical and virtual manipulatives; 3) using diagrams, models, pictures, and problem-solving techniques; and 4) using child-friendly vocabulary.

The first theme in the literature related to MTEs providing the opportunity to develop PTs' knowledge of content and teaching was largely represented by studies that suggested *implementing worthwhile tasks* (e.g., rich, engaging, meaningful, and mathematically ambitious) that PTs could facilitate in a K-8 classroom (e.g., Kuennen & Beam, 2020; Feldman et al., 2016; Feldman, Wickstrom, Ghosh Hajra, & Gupta, 2020; Fung & Latulippe, 2010; Li & Castro Superfine, 2018; Litster, MacDonald, & Shumway, 2020; Hallman-Thrasher, Rhodes, & Schultz, 2020). *Worthwhile Mathematical Tasks* is the first standard identified in NCTM's *Professional Standards for Teaching Mathematics* (1991). Furthermore, this strategy has been a recommendation for effective professional development for teachers of mathematics, for nearly 20 years (e.g., Loucks- Horsley et al., 1998). For example, Fung and Latulippe (2010) described an activity called "newspaper headlines," where PTs are given quantitative data from headlines and quotes from the newspaper. The PTs were asked whether the number was precise or whether there was a possible error somewhere in the reporting. This activity provided the opportunity for PTs to attend to estimation, precision, and the usefulness of estimation in the world—potentially leading them to emphasize number sense in their future elementary classrooms.

In their study on MTEs' perceptions of content course design for elementary PTs, Li and Castro Superfine (2018) asked their participants how they achieve their course design goals. *Implementing worthwhile mathematical tasks* was one of the two most common instructional methods that their six participants articulated. Furthermore, the MTEs stated that they focused on tasks that are hands-on and/or those that allow for problems to be solved in multiple ways.

Hallman-Thrasher, Rhodes, and Schultz (2020) refer to explanation-worthy tasks that are tasks for PTs which “provide opportunities for PTs to explain and justify their thinking about important mathematical concepts and procedures *and* consider ways that students think about mathematics” (p. 886). They emphasize that these tasks should: a) challenge and engage PTs while aligning to grade K-8 mathematical content, b) challenge PTs concerning what it means to know and do mathematics, c) challenge PTs’ mathematical misconceptions, and d) provide PTs with the opportunity to explore K-8 student thinking and misconceptions. Also, Litster et al. (2020) suggest that MTEs use tasks that promote reasoning and problem solving in order to facilitate an active learning environment in mathematics content courses.

A second theme found in the literature was *using physical and virtual manipulatives* in content courses for PTs (e.g., Appova & Taylor, 2019; Kuennen & Beam, 2020; Darley & Leapard, 2010; Johnson & Olanoff, 2020; Li & Castro Superfine, 2018). Engaging PTs with manipulatives in content courses can help PTs make their own mathematical connections among the concepts under discussion and also help to reduce their math anxiety (Karunakaran, 2020). Furthermore, using manipulatives that are used to teach K-8 mathematics may help PTs realize how such tools can help further the learning of their future K-8 students. For example, one MTE in Li and Castro Superfine’s (2018) study stated that she engaged her PTs in hands-on tasks in order to model the “beyond memorization” knowledge development in their future elementary classrooms. She stated,

> I try to make it (task) hands on as much as possible so that the properties and ideas come from what they’ve done because I think they (preservice teachers) connect (mathematics concepts) better; they can hold on to it. So I take a lot of what’s in the book and I try to make it hands on. It’s probably the best I could say: What are the differences between an inch and a centimeter? What is the connection? They memorized it but… You’re supposed to show a kid what the difference is, but not just tell them. Here is an example. The right triangle, 30, 60, 90, and the special right triangles, 45, 45, 90. We do the paper

> folding. They (preservice teachers) find what these sides are and generate the ratios, and then they do problems about these triangles. (p. 191)

In this example, the authors did not specifically mention they were providing the opportunity for their PTs' to develop knowledge of content and teaching, however, here, the MTE is engaging the PTs in developing concrete teaching strategies they could use to teach mathematics to students.

Appova and Taylor (2019) also reported a similar example of an MTE engaging her PTs with K-8 manipulatives. In fact, in their study, all MTEs articulated an instance implementing opportunities for PTs to learn about K-8 manipulatives as instructional tools. One MTE shared that she "set up four stations and each station had Base-10 blocks. The [PTs] had some problems to do but they had to model it with the blocks when they went [to the stations]" (p. 11). Similarly, Darley and Leapard (2010) provided their PTs with opportunities to connect algorithms to physical models, in order for PTs to be able to better explain their reasoning and use K-8 mathematical language and connections. The authors elaborated that the PTs had "to explain concepts to others using many different representations," including manipulatives (p. 186). Furthermore, the fact that the emphasis was on giving explanations to help others make connections between physical models (i.e., using manipulatives) and the mathematical concept (e.g., partitioning [sharing] interpretation for division), the PTs observed they now had a clear grasp of the topic and new strategies to teach this topic to students.

Johnson and Olanoff (2020) posed a task where PTs were asked to not use procedures to solve a proportional reasoning task that involved mixtures. The manipulatives the MTEs provided was a tool the PTs used to help them build their conceptual understanding of the meaning of ratios and proportions.

A third theme identified in the literature was to provide PTs the opportunity to *use diagrams, pictures, and problem-solving techniques*, so they have the opportunity to represent mathematics in a meaningful way and develop a deeper understanding of K-8 mathematics for themselves. In alignment with this third theme, Veselovsky (2017) studied four MTEs who taught mathematics content courses. Several of the MTEs articulated that they provide PTs with the opportunity to draw pictures in order to help them use pictorial representations as a bridge between the concrete and symbolic representations of a mathematical concept under discussion. Likewise, Nickerson and Whitacre (2010) conducted an experiment in a content course where multiple problem solving techniques served as *models of* PTs' thinking (e.g., nice numbers, reasonable jumps on a number line, shifting the difference) to develop mathematical understanding of a *model for* an empty number line where the minuend and subtrahend were both represented as locations on the number line. The authors further articulated,

> The shift from *model of* [multiple problem solving techniques] to *model for* [number line] describes a transition from a phase in which students' strategies are deliberately represented with the model to one in which reasoning with the model enables students to make sense of the mathematics in new ways, which may lead to the invention of new model-based strategies. (p. 245)

A final strategy identified by the literature was where MTEs *use child-friendly vocabulary* to help PTs "become more familiar with the appropriate language for communicating mathematics to K-8 students". One of the MTEs in Appova and Taylor's (2019) study provided an example from his content course,

> We talk about polyhedra and non-polyhedra, but in kindergarten you may not use that language. So, how do you get that idea across without the word? I make them [PTs] start thinking about that. I make them think about what is a kid going to call a vertex? So talking about the language kids use and trying to build on their language at appropriate grade levels. (p. 12)

In this example, the MTE highlights the need for PTs to choose appropriate vocabulary, that may not have been introduced yet, to help K-8 students make connections to the topics that they are learning or will be learning in the future classes.

This idea of developing the language of mathematics as an essential aspect of teaching mathematics to young students is prevalent in the practitioner literature for in-service teachers and is critical for preparing PTs as well (e.g., Cuevas, 1984; Monroe & Orme, 2002; Riccomini, Smith, Hughes, & Fries, 2015). Some strategies include: a) use advanced organizers and concrete props (e.g., Herrell & Jordan, 2008), b) develop students' vocabulary by identifying words that may cause confusion for students in a lesson (e.g., Khisty, 1997; Torres-Velasquez & Lobo, 2005; Weist, 2008), and c) use visual cues such as diagrams and photographs and write the spoken words on the blackboard to minimize the amount of time students spend just listening (e.g., Khisty, 2002; Torres-Velasquez & Lobo, 2005; Weist, 2008). These suggested strategies support mathematical instruction for all students.

The four themes identified above shed light on prevalent MTE practices, in the literature, associated with providing the opportunity for PTs to develop PCK. Furthermore, new research is also being published identifying the ways PTs develop their knowledge bases with respect to PCK. Specifically, in their study, Chick and Beswick (2017) proposed a PCK framework identifying several practices that MTEs can employ to engage PTs in developing both knowledge of content and teaching, including how to use concrete materials to demonstrate a [mathematical] concept and describe or demonstrate ways to model and illustrate a [mathematical] concept, including the use of pictures and diagrams. The findings of Chick and Beswick (2017) about PTs' PCK development are closely aligned with the extant literature on MTEs' PCK-related practices. Furthermore, the authors proposed recommendations for the ways

PTs develop PCK knowledge domains to provide a theoretical and practical lens for MTEs to be able to closely reflect upon the events of their practice and observe specific evidence of PTs' knowledge existence and development in their courses.

Opportunities to Develop Prospective Teachers' Knowledge of Content and Curriculum

Curriculum theorists distinguish among different meanings and categories of *knowledge of curriculum*. For example, knowledge of "intended" curriculum refers to the understanding of standards and curriculum frameworks outlined in the national, state, and local policies. And, the knowledge of "textbook" curriculum is comprised of teachers' experience with and expertise in the pedagogical resources/materials specific to teaching and learning of school mathematics, including the scope and sequence of specific (grade-level) topics included in the textbook (Remillard, 1999; 2005). However, the "intended" and "textbook" curricula include key instructional goals and pedagogical frameworks/philosophies, which may be quite different from what actually transpires in the classroom (i.e. "enacted" and "learned" curriculum [Gehrke, Knapp, & Sirotnik, 1992; Remillard, 2005]). Since content courses typically do not include a field-placement component and PTs are not required to teach K-8 students as part of their content courses, this section is focused on ways MTEs provide the opportunity for PTs to develop *knowledge of curriculum*—primarily the "intended" and "textbook" curriculum.

Opportunities to develop prospective teachers' knowledge of "intended" curriculum. The "intended" curriculum for school mathematics has evolved in the past three decades. In 1989, NCTM published the Curriculum and Evaluation Standards, which proposed the Standards for school mathematics to include a core curriculum for all students. The Standards launched a reform-movement in the United States, followed by multiple Standards documents put forth by the NCTM (e.g., 1989; 1991; 1995). These led to the Principles and

Standards for School Mathematics (NCTM, 2000), which reformulated the teaching of mathematics through connections, representations, problem solving, and modeling, and added the content of statistics, probability, and discrete mathematics to middle school grades.

These curriculum initiatives served as the catalysts for nation-wide reform efforts to redefine K-12 mathematics teaching and learning (Ball & Feiman-Nemser, 1988; Burstein, 1993). Today, every state has coherent and comprehensive *curriculum* standards, with the majority of states having adopted the *Common Core* (CCSSI, 2010) as their state standards (Friedberg et.al., 2018; Peterson & Hess, 2008). These efforts are aimed at providing mathematics teachers with better-quality curriculum to assist in planning and teaching, including "scope and sequence" recommendations to help better support students' learning (Kuhs & Freeman, 1979; Weiss, Banilower, McMahon, & Smith, 2001). The *Standards* explicitly advocate to focus mathematics instruction on moving students beyond the surface-level rote and algorithmic procedural knowledge and working towards the development of a deeper conceptual knowledge and problem-solving skills (Schmidt & Houang, 2012). Thus, MTEs who prepare future teachers have a responsibility to articulate and elucidate the *Standards* and the educational reform efforts to their PTs in content courses (for more information on the evolution of national content and process *Standards* and how they have informed recommendations for teacher preparation see Max & Welder [2020]).

There are several ways, documented in the literature, to do this. Most of them involve MTEs (skillfully) putting PTs in a position to directly reflect upon their beliefs and ideas about mathematics teaching and learning, as well as educational reforms, and situating those discussions around PTs' short-term goals (e.g., learning a specific mathematical concept/topic) and long-term goals of becoming a teacher (Jansen, 2009). For example, Bernander, Szydlik, and

Seaman (2020) offer various suggestions and classroom based strategies on how to incorporate opportunities for PTs to learn about Common Core Mathematical Practices, including specific vignettes and examples that help to foster discussions about curriculum as well as specific learning expectations addressed in the *Standards*.

Many educators also suggest that MTEs use media as an impetus for discussions about K-12 educational reforms, as media can help to keep them from becoming personal, and support MTEs in facilitating productive discussions, particularly for MTEs who do not have the experience to draw upon from their K-12 teaching or to critique divergent viewpoints related to K-12 educational reforms (Appova & Taylor, 2019; Chval, Lannin, Arbaugh, & Bowzer, 2009; Meyer, 2015; Sztajn, Ball, & McMahon, 2006). Different media sources (e.g., blogs, videos, talks) are available specifically addressing curriculum-related issues, including: a) the need for more mathematical modeling and reasoning opportunities in textbooks and school curriculum (e.g., Meyer, 2010), b) the importance of inquiry-based and problem-solving approaches to mathematics teaching and the positive impact of these approaches on students' learning and motivation (e.g., Boaler, 2014; Finkel, 2016), and c) the need for evolution, development, and adoption of mathematics education *Standards* (Jones, 2017; Lasevoll, 2018). In regard to implementation of the media resources during class, many studies suggest posing specific questions for PTs' individual and collective reflections and discussions.

For example, Chval, Lannin, Arbaugh, and Bowzer (2009) reported using a public media piece in their classes to inspire discussions about the political nature and importance of curricular changes in mathematics education. The authors showed a video (*Prime Time Live* with Diane Sawyer, 1998), which included interviews with prominent scholars about the results of the international studies for (mathematics and science) comparisons between the US students and their foreign counterparts (Trends in International Mathematics and Science Study [TIMSS]; Programme for International Student Assessment [PISA]). To make these discussions effective, the authors encouraged PTs to consider the statement made in the video by a mathematics educator, Bill Schmidt, "we really do teach mathematics so differently from the rest of the world." The authors also asked PTs to consider "the viability of the problem-solving instructional strategies employed in other countries" and to respond to the following prompts (Chval et al., 2009, p.103): "According to the researchers [in the video], what does typical American math teaching and textbooks look like and how do they differ from the rest of the world?"

Similarly, Appova and Taylor (2019) found that *expert* MTEs, when teaching content courses, include these types of discussions to motivate PTs' curiosities and interests regarding the political climate and public state of mathematics education. In their study, MTEs shared that PTs often brought to class a public opinion piece, found on social media (e.g., parenting blog and learning blog of New York Times), asking the MTEs to help make sense of the public pushback and criticism on the reform educational initiatives. One MTE shared that posing a few reflection questions back to the PTs was an effective strategy in her content course. She described,

> Last year we talked about Common Core because it was in the news so much. You know, that one dad who sent back homework. Some one of my students [PTs] brought it up and I was glad that it came from them. My response was, "I don't care what you call it, No Child Left Behind, Common Core, the next thing, whatever… Why does this dad think that this is bad for his child? Looking at this homework, what do you see? Does it help the child learn to think? Learn mathematics better?" Many of them [PTs] thought about it, I heard some say, "I don't know why that dad was so upset. It's really building critical thinking. It's helping the child analyze, and you're really getting to those higher levels of thinking in the Bloom's taxonomy." We also talked about Bloom's taxonomy. (p. 194)

PTs' familiarity with and understanding of the state and national curriculum *Standards* is critical. In fact, Gurl, Fox, Dabovic, and Leavitt (2016) argue that, "Just as teachers are required to support their own students in persevering in problem solving, those who prepare teachers must support them in persevering to meet the Standards for Mathematical Practice" (p. 34). Gurl and colleagues reported that, in their courses, they facilitate PTs' analyses of the *Standards for Mathematical Practice* (CCSSI, 2010) through "mini-lessons" that PTs teach as their peers role-play as school students.

Appova and Taylor (2019) found that *expert* MTEs dedicate a portion of their class to discussing different curriculum documents (e.g., CCSSI, 2010; NCTM, 2000; NRC 2001) and do so either by directly using/sharing these documents or selecting a few focal points for PTs to reflect upon. In their study, MTEs used *Standards* documents: (1) to familiarize PTs with curriculum standards, and (2) to help them better understand why their own learning in the content courses is structured in the way that directly mirrors these standards. Furthermore, the MTEs wanted PTs to know the grade-level scope and sequence included in the *Standards* and when the mathematical concepts are first introduced to K-8 students. For example, one MTE elaborated on his discussions with PTs about division of fractions in the K-8 curriculum,

> We talk about how the content we're covering [in the content courses] relates to what the [K-8] students have to do. For example, we talk about how modeling the division of fractions is actually something that appears in 5th grade, so they [kids] are going to be asked to do these things that we are doing in class. (p. 194)

This MTE articulated that examining the scope and sequence of the K-12 curriculum helps PTs to better understand the ways (and grade-levels) in which different mathematical ideas develop and evolve over time. The MTEs also indicated that they try to explicitly connect these K-12 topics to their content courses and encourage PTs' appreciation for studying them at greater depths (2019).

Opportunities to develop prospective teachers' knowledge of "textbook" curriculum. Research shows that novice K-12 mathematics teachers use textbooks in a distinctly different way when compared to experienced teachers (Brown & Edelson, 2003; Christou, Eliophotou-Menon, & Philippou, 2004; Remillard & Bryans, 2004; Sherin & Drake, 2009). For example, Sherin and Drake (2009) found that before teaching each lesson, experienced teachers evaluate and adapt their textbooks to their students' needs, whereas novice teachers mainly read the textbook noting the "details," rather than "big ideas." These findings were further supported by another study, documenting that beginning teachers, when implementing new textbooks, primarily were self- and task-focused, as opposed to focusing on their students' needs as demonstrated by experienced teachers (Christou, Eliophotou-Menon, & Philippou, 2004). These studies strongly suggest that PTs need to gain experiences with elementary mathematics textbooks and resources in their teacher preparation programs, particularly to develop a better understanding about the role of the textbooks for planning and teaching.

Erb (1991) argued that "if teacher education is to contribute to breaking the inertia of curricular tradition, then programs must expose PTs to the characteristics of curricular organization that are unique to the elementary and middle grades" (p. 25). Opportunities for doing so found in the literature included having PTs: a) explore different (e.g., local, national, international) textbook materials (e.g., Reys, Reys, Lappan, & Holliday, 2003; Senk & Thompson, 2003); b) examine textbook variability in terms of teaching philosophies and/or curriculum structures (e.g., Chingos & Whitehurst, 2012; Hjalmarson & Suh, 2008; Stein, Remillard, & Smith, 2007); and c) investigate textbook resources in regards to their mathematical rigor, scope, and sequence, and how mathematical concepts are presented and developed across different grade levels (e.g., Chávez, Tarr, Grouws, & Soria, 2015; Lutz & Berglund, 2007; Schmidt & Houang, 2012; Weiss, Banilower, McMahon, & Smith, 2001).

For example, Hjalmarson and Suh (2008) described PTs evaluating different curricular materials and resources (e.g., math applets, lessons, textbooks) in regard to their teaching approaches, curriculum structure, and organization. The authors found that PTs not only deepened their knowledge about mathematics textbooks (and resources), but they also developed an ability to recognize and focus on the "beyond-surface" level learning opportunities that may result from the use of those textbooks. Likewise, Tirosh (2000) documented that engaging PTs in discussions and investigations of various fractions-related activities found in different elementary and middle school textbooks helped to strengthen PTs' curricular knowledge and improve their mathematical knowledge about rational numbers. Authors from several additional studies also reported that engaging PTs in tasks modified from mathematics elementary textbooks, and focusing their attention on both problem-posing and problem-solving strategies, helped to illuminate different aspects of PTs' mathematical knowledge of the content and deepen their reasoning strategies (e.g., Thanheiser et al., 2016; Tobias et al., 2014).

The CBMS (2001) strongly suggests that teacher preparation courses help PTs "make connections between the mathematics being studied [in their courses] and mathematics prospective teachers will teach" (p. 7). To address these recommendations, Lutz and Berglund (2007), in their content courses, adopted and used several mathematically rich lessons from middle school textbooks. They articulated,

> When choosing K-8 textbooks to use in the content courses, we thought about PTs achieving a richer view of mathematics, seeing it as a discipline that involves logic and reasoning rather than simply memorization… The PTs were challenged by the [K-8 textbook] material. They learned new mathematics and gained a deeper understanding of mathematics they had already been taught. They developed perspectives more in line with current thinking in mathematics education. Teaching mathematics to PTs should be qualitatively different from the way mathematics has traditionally been taught. (pp. 346-347)

Opportunities to Develop Prospective Teachers' Knowledge of Content and Students†

Knowledge about students' mathematical thinking and learning is considered one of the most critical domains of PCK (Ball et al., 2008). Teachers must be able to anticipate what students are likely to think, do, and find confusing with mathematics content (Ball et al., 2008). When assigning a task, teachers need to anticipate what students are likely to do with it and whether they will find it easy or hard. Teachers must also be able to hear and interpret students' emerging and incomplete thinking. Being able to do this requires an interaction between specific mathematical understanding and familiarity with students and their mathematical thinking.

We found, in the current literature, that MTEs engage in different practices situated around PTs' development of knowledge of content and students, which mainly involve two types of activities: *direct* and *indirect* interactions with students (e.g., Ball et al., 2008; Carpenter, Fennema, Franke, Levi, & Empson, 1999; Hill et al., 2008). Specifically, some MTEs provide opportunities for PTs to visit local schools and work *directly* with students via live observations of learning, interviews, and administration of mathematical tasks (e.g., Appova & Taylor, 2019; Fernandes, 2012; Johnson, Campet, Gaber, & Zuidema, 2012; Stephens & Lamers, 2006). In contrast, *indirect* opportunities may include PTs observing and analyzing students' thinking and learning by watching videos and examining authentic artifacts collected from students (e.g., work samples, written solutions). Below, we discuss these learning opportunities as course activities that involve *direct* and *indirect* PTs' interactions with students.

Opportunities to develop prospective teachers' knowledge about students via direct student interactions. One of the main themes identified in the literature was for PTs to engage in interviews with school age students, which has been shown to strengthen PTs' mathematical

†Some portions of this section, *Opportunities to Develop Prospective Teachers' Knowledge of Content and Students*, have been previously published in Appova (2018).

and pedagogical knowledge related to students' problem-solving strategies and mathematical thinking (Fernandes, 2012; Friel, 1998; Gee, 2006; Jenkins, 2010; Lannin & Chval, 2013; McDonough, Clark, & Clark, 2002; Spangler & Hallman-Thrasher, 2014). Specifically, studies show that interviews help PTs to develop awareness of the variety of problem-solving/thinking strategies that students used when doing mathematics and engender PTs' ability to adopt an interpretative rather than evaluative perspective when analyzing students' solutions (Crespo, 2000; Mason, 2002). However, to be effective, student interviews must include a clear and focused objective, purpose, and structure.

For example, when discussing with PTs their conceptions (and misconceptions) about statistics and measures of central tendency in her content course, Friel (1998) engaged PTs in a four-task interview to help them determine elementary students' understanding of the concept of average. The interviews involved low- and high-level tasks. PTs were given a specific protocol to follow that included: a) interview a small group of students (individually) about their thinking as they try to solve a mathematical task, and b) listen to and keep records of their strategies. Spangler and Hallman-Thrasher (2014) in their concurrent methods course, taught alongside a content course, reported PTs working directly with students by posing mathematical problems to the students while observing and gaining insights into their thinking about number and operations concepts. Both studies reported improvement in PTs' mathematical and pedagogical knowledge from working directly with students. Furthermore, in her concurrent content course, taught alongside the methods course, Gee (2006) described the "Math Mates" structured interviews, where PTs focused their attention on students explaining their solutions while PTs asked follow-up questions to probe student thinking. PTs were provided with specific follow-up questions to ask (e.g., *How did you get that answer? Can you show me how you solved that problem?*).

Johnson, Campet, Gaber, and Zuidema (2012) reported that virtual manipulatives in their content courses also helped PTs to conduct student interviews. Likewise, these MTEs provided PTs with specific "manipulatives-based" follow-up questions to ask students during the interviews (e.g., "Can you use the pieces on the screen to show me what you are doing/thinking? Tell me why you are [shading, moving that part, selecting that portion] of the object? How does that help you solve the problem?" [p. 203]). The authors also reported that clinical interviews helped to improve PTs' skills in carefully selecting mathematical tasks, and their corresponding manipulatives, and helped PTs to develop meaningful prompts to elicit students' responses.

Furthermore, many studies documented non-interview opportunities to engage PTs in fruitful *direct* interactions with students, including giving written assessments, prompts, or questions for students to respond (e.g., Sjoberg, Slavit, & Coon, 2004; Stephens & Lamers, 2006). These may include take-home or in-class prompts, quick problems or larger projects, quizzes, and exit slips. Studies show that administering assessments help PTs gain insights into K-8 students' thinking and gauge their prior and existing knowledge as well as gaining a better understanding of how the nature of the questions/prompts influence the information gained from the student (e.g., Sjoberg, Slavit, & Coon, 2004; Stephens & Lamers, 2006).

For example, Stephens and Lamers (2006) created specific guidelines that helped their PTs develop assessments: "Does the task specifically assess your chosen content area and the conceptual understanding of the content, rather than merely procedural skill? Does it elicit different representations/strategies? Will students' responses provide you with valuable feedback about their thinking?" (p. 119). Similarly, Sjoberg, Slavit, and Coon (2004) argued that "it was helpful to convey [to PTs] clear expectations, demonstrate effective writing in mathematics, and provide clear and specific prompts" for PTs to help them better elicit students' thinking (p. 490).

Authors of several studies also suggest that, in mathematics content courses, engaging PTs in after school projects (e.g., family "math nights") is also an effective way for PTs to develop the knowledge of students' mathematical learning (e.g., Bofferding, Kastberg, & Hoffman, 2016; Freiberg, 2004; Lachance, 2007; Lachance, Benton, & Klein, 2007; Shokey & Snyder, 2007). These experiences often involve different groups (e.g., students, their families, teachers, administrators, staff, college instructors), however, the primary goal is for these groups to have an opportunity "to learn about mathematics together in an informal and supportive setting" and gain a better appreciation for "how mathematics is integrated into a variety of contexts and subject areas while having fun" (Lachance, 2007, p. 407). Popular "math night" activities involve stations, where each pair of PTs may have an activity table structured in a form of a contest/game for students to play/compete against themselves, their parents, or the PTs (Freiberg, 2004; Lachance, 2007; Lachance, Benton, & Klein, 2007).

Afterschool and pen-pal projects are also gaining popularity as they allow PTs and students to engage in mathematics with the purpose of helping students complete a mathematically rich activity (individually or in small groups). For example, Shokey and Snyder (2007) described an afterschool "tessellation t-shirts" activity, where K-4 students (with the help of PTs) repeatedly traced their tessellating designs on a large piece of newsprint and colored them with fabric crayons. "Such designs can be ironed onto T-shirts for great personal mathematical artwork. The college instructors and PTs operated the ironing stations" (Shokey & Snyder 2007, p. 86). Engaging in this activity enhanced PTs' knowledge about elementary students' conceptions (and misconceptions) related to geometric patterns and tessellations.

As pen-pals, PTs and school students exchange "math letters" related to rich mathematical problems, for which students received personalized feedback from PTs on their solutions and learning. Lampe and Uselman (2008) reported PTs making meaningful connections with their pen-pal students in a "quasi-teaching capacity," which allowed them to put their knowledge into practice, and students put "greater-than-usual" effort in developing their solutions as well (p. 200). Crespo (2003) argued that pen-pal activities provided the opportunity for PTs to develop knowledge of content and students by encouraging her PTs and their young counterparts "to make explicit their mathematical thinking, deepen their own understanding of the subject, and become more comfortable doing and talking about mathematics" (p. 34).

Overall, suggestions from the literature highlight that engaging PTs in *direct* student interactions help them gain experiences in working with students, understanding student thinking, and making meaningful mathematical connections between students' thinking and mathematical topics. Lannin and Chval (2013) call these opportunities "powerful" as they provide PTs with firsthand insights into students' learning and embed opportunities for PTs to try out various instructional strategies to address students' misconceptions. Most importantly, these opportunities allow PTs to recognize how difficult it is to gain insight into students' thinking, particularly the challenge of selecting a meaningful task or a question, and be able to draw accurate conclusions about students' knowledge (Lannin & Chval 2013).

Opportunities to develop prospective teachers' knowledge about students via indirect student interactions. If direct access to school students is not viable in mathematics content courses, many other opportunities are available for PTs to examine and analyze students' thinking and learning indirectly, such as using videos and authentic artifacts collected from students (e.g., work samples, excerpts, written solutions). Extensive databases and resources are available offering videos, transcripts, and case studies of students (e.g., Cognitively Guided Instruction [CGI]; Integrating Mathematics and Pedagogy [IMAP]; Number Talks [e.g., Parrish, 2011]; Show-Me Center; Teaching Channel). Also see Max and Welder (2020) for additional resources that could be shared with PTs. Many of these projects offer recommendations on how to use their resources with teachers. For example, the CGI project offers guidelines on how to use their videos and materials to help in-service and prospective teachers learn about specific problem types (whole number operations) and different strategies and levels of thinking that elementary students use to solve those problems (e.g., Carpenter, Fennema, & Franke, 1996).

Research also shows that using audio and video analyses with PTs helps to develop their *noticing* skills when examining students' mathematical thinking and learning (e.g., Amador, 2017; Carpenter et al., 1999; Castro Superfine, Prasad, Welder, Olanoff, & Eubanks-Turner, 2020; Jilk, 2016; McDuffie et al., 2014). For these examinations, the *professional noticing* framework is often used to provide critical lenses for PTs to effectively analyze the videos and students' work samples and make sense of them via "attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings" (Jacobs, Lamb, & Philipp, 2010, p. 172; also see Philipp, 2008; Sherin & van Es, 2003; Thomas et al., 2015).

McDuffie and colleagues (2014) used video analysis to support PTs' *noticing* of children's multiple knowledge bases and their levels of thinking in an elementary mathematics content course. Similarly, Amador, Estapa, de Araujo, Kosko, and Weston (2017) used the *noticing* framework in their courses to "purposely focus on [children's] mathematical content" to elicit and analyze features to which PTs attend to students' understanding (p. 158). Specifically, Amador et al. reported utilizing videos and animated video cases of students' learning to help

PTs *notice* the mathematical content related to students' fractional understanding. They engaged PTs in discussions about *who* (students and teacher) and *what* (mathematical concepts and pedagogical moves) they noticed while watching the videos.

Several studies also recommend video cases to help promote PTs' learning and awareness of the issues related to equity, social justice, and responsive pedagogy. Jilk (2016) reported using videos to encourage PTs to look for and pay closer attention to the resources (what students have) and potentials of students, rather than their deficits. When teachers learn to notice and focus on students' strengths, they begin to perceive and approach them as competent learners (Cohen, 1994) and, through those observations, teachers are able to support student development of positive mathematics identities (de Abreu & Cline, 2006; Jilk, 2014; Martin, 2000, Nasir, 2002). These aspects of teaching are critical and necessary for creating robust opportunities to learn for *all* students, particularly "for young people who have traditionally been marginalized by school mathematics" (Jilk, 2016, p. 188).

In addition to the videos, MTEs may also implement students' written work and solutions to provide PTs with opportunities to examine students' thinking. For example, Herbel-Eisenmann and Phillips (2005) assigned sixteen algebra problems for PTs to solve and sort based on their characteristics. After sharing their own solutions, PTs were presented with students' work (from the same problems) and asked to articulate what they noticed about students' mathematical understandings in the problem. The authors also reported that examining student work helped their PTs to become "more aware of the mathematics embedded in the problem and the diversity of students' algebraic reasoning" (Herbel-Eisenmann & Phillips, 2005, p. 63). Similarly, Cianca (2013) reported that by analyzing students' drawings of geometric figures (e.g., pictorial, schematic, isometric) PTs not only increased their own knowledge of geometry but also strengthened their *noticing* skills for assessing students' work and critiquing their solutions. In the areas of number and operations, studies have documented that students' work with number arithmetic can be quite powerful and useful in highlighting students' unconventional and/or more conceptual thinking. For example, Castro Superfine et al. (2020) described instances where PTs, who only understood the traditional subtraction algorithm, struggled to make sense of a student's nontraditional, albeit correct, work and thinking. PTs recognized the limitations of their own understanding of subtraction and how it may limit their ability in meeting the needs of their future students. Furthermore, these instances helped PTs to recognize that they may not know enough mathematics to effectively teach it to elementary students and supported their motivation to want to learn more.

Furthermore, analyzing students' errors has been reported to help PTs make better sense of students' struggles, reason about their misconceptions quantitatively, and critique their mathematical problems (see Borasi, 1994), which are three of the eight *Standards for Mathematical Practices* (CCSSI, 2010). Several resources are available in the field that provide examples of students' work involving common mathematical error patterns (see Ashlock, 2001). For example, Lim (2014) reported that analyzing and discussing students' errors in work samples offered PTs an opportunity to tackle certain misconceptions "head-on, learn from those mistakes, discuss the mathematics underlying those errors, and deepen their own mathematical understandings" (Lim, 2014, p. 111). The author also offered specific characteristics that can help MTEs identify "error-eliciting" work to discuss with PTs, including students' solutions that bring forth "common" mistakes and elicit misconceptions, misapplication of a procedure or a formula, and overgeneralization of a concept.

Ultimately, activities in a mathematics content course that (directly or indirectly) focus PTs' attention on students' mathematical thinking and learning help PTs become better prepared for field experiences and future teaching careers. These activities are particularly effective at strengthening PTs' abilities "to anticipate students' responses, to redirect partially correct and incorrect responses, and to match follow-up questions and suggestions to the students' thinking" (Spangler & Hallman-Thrasher, 2014, p. 63).

Conclusion

Researchers argue that increasing and deepening PCK is critical for the development of PTs' professional identities and readiness to teach (An et al. 2004; Blomeke et al. 2014; Marks 1990). In fact, An et al. (2004) state that enhancing PCK "should be the most important element in the domain of mathematics teachers' knowledge" (p. 146). However, research also shows that often, when teaching mathematics content courses to PTs, university instructors predominantly focus on subject matter knowledge and not enough on the elements of PCK, and that many instructors may not have expertise and teaching experience in embedding PCK-related learning opportunities into their mathematics content courses (Ball et al., 2008; Hill et al., 2008; Schmidt, Burroughs, Cogan, & Houang, 2017; Laursen, Hassi, & Hough, 2015; Sztajn, Ball, & McMahon 2006).

In this article, we drew from the literature to offer specific classroom-based examples of how other MTEs structure learning opportunities in their content courses to help develop PTs' PCK. By conducting this literature review, we offer the field an account for evidence demonstrating that all MTEs, regardless of their teaching experience or expertise in working with PTs or schoolchildren, have readily available classroom-based examples and professional literature that can guide them in implementing activities in their content courses to help develop

PTs' PCK. Furthermore, we offer the field specific research accounts that comprise evidence showing that these course activities indeed improve PTs' readiness to teach mathematics and their ability to better support students' learning and academic development. This is particularly due to the intertwined nature of the PCK components.

In fact, in our literature review, we uncovered a strong consensus across the studies suggesting that the knowledge that matters in teaching (and for preparing teachers) is deeply rooted in the activities that teachers perform daily, and those activities often involve the use of all PCK components concurrently. Ball (2000) described these activities to include, "figuring out what students know [knowledge of content and students]; choosing and managing representations of ideas [knowledge of content and teaching]; appraising, selecting, and modifying textbooks [knowledge of content and curriculum]; and deciding among alternative courses of action, and analyze the subject matter knowledge and insight entailed in these activities [knowledge of content and teaching]" (p. 244). The activities we found in the literature (see Table 1) directly align and echo this notion. We believe they will help MTEs better prepare PTs.

The collection of PCK-related activities described in this paper may not be exhaustive, and their implementation and applicability may vary across different settings and teacher education programs; however, the work here provides a foundation for MTEs' practice encompassing classroom-based examples and strategies and offers potential avenues for MTEs to research the nature and effectiveness of their own course activities that may help develop PTs' PCK. Furthermore, we hope that when designing learning opportunities for PTs in content courses, MTEs keep at the forefront developing their PTs' PCK, as well as the PTs' deep and insightful understanding of mathematics taught at the K-8 grade levels.

Table 1

Synopsis of MTEs' Practices and Content-course Activities Found in the Literature for Providing Opportunities to Develop K-8 Prospective Teachers' (PTs) PCK

PCK Component	Literature Associated with PCK Component
Opportunities to Develop Prospective Teachers' Knowledge of Content and Teaching	Implementing *worthwhile tasks* (e.g., rich, engaging, meaningful, and mathematically ambitious) that PTs could facilitate in a future K-8 classroom. Providing PTs with multiple opportunities to use (and learn to use) *pictures, diagrams, models, and various problem-solving techniques* as part of their mathematical skills and repertoire of pedagogical strategies for teaching K-8 mathematics.
	Using physical and virtual *manipulatives* that have direct relevance for teaching K-8 mathematics and are appropriate for use with K-8 students when teaching mathematics.
	Using (and learning to use) *child-friendly vocabulary* to help PTs become familiar with the appropriate language for communicating mathematics more effectively to K-8 students as well as strategies for developing students' *mathematical language* and vocabulary.
Opportunities to Develop Prospective Teachers' Knowledge of Content and Curriculum	Embedding various *professional resource*s (e.g., state standards, readings from literature, professional talks) and media sources (e.g., blogs, posts, videos, news, newspaper articles) to prompt PTs' knowledge of intended curriculum and to inform, challenge, and expand their knowledge and beliefs about mathematics teaching and learning, and educational reforms.
	Integrating analyses of *textbook curriculum* to help PTs develop a better understanding of the structure, organization, rigor, scope, and sequence of mathematical topics in these textbooks, as well as their understanding of the differences in the teaching approaches and philosophies (in textbooks) and how those differences may impact students' learning outcomes.
Opportunities to Develop Prospective Teachers' Knowledge of Content and Students	Engaging PTs in experiences that offer *direct interactions* with school aged students (e.g., interviews, mini-lessons, quick assessments, family nights, pen-pal projects) or *indirect interactions* using artifacts collected from students (e.g., videos, work samples, excerpts, written solutions) to help develop PTs' knowledge of students' mathematical thinking and problem-solving approaches and develop pedagogical strategies to effectively analyze and address students' responses, solutions, and misconceptions.

In conclusion, we argue that it is critical for PTs to be provided with opportunities to observe and learn from MTEs who are equipped with vivid and descriptive examples as well as models of strategies that reflect and support the types of mathematical learning and teaching methods valued by the profession (Ball et al., 2009; Feimen-Nemser, 2001; Ghousseini & Herbst, 2016; Sherin, 2001). We hope the examples and recommendations provided in this paper help MTEs explore professional resources to target, address, and deepen PTs' PCK. Furthermore, we hope the examples provided here help MTEs to recognize how different the endeavor of teaching mathematics to PTs may be, especially when compared with teaching mathematics to other college majors or K-12 students (Ball et al., 2009; Zeichner, 2005).

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